Lecture 5 - January 24

Math Review

Relations

Announcement

- Lab1 submission due in a week
  - + tutorial videos
  - + problems to solve
  - + Study along with the Math Review lecture notes.

Sets: Exercises

$4 \leq 7$ T
$\wedge$
$4 \geq 7$ F

Set membership: Rewrite $e \notin S$ in terms of $\in$ and $\neg$

$e \notin S \equiv \neg(e \in S)$

Find a common pattern for defining:

1. $=$ (numerical equality) via $\leq$ and $\geq$ $\rightsquigarrow$ $\forall x, y \cdot x \in \mathbb{Z} \wedge y \in \mathbb{Z} \wedge x = y \Rightarrow x \geq y \wedge x \leq y$.
2. $=$ (set equality) via $\subseteq$ and $\supseteq$

$S = T \Rightarrow \boxed{S \subseteq T \wedge S \supseteq T}$.

$S = \{1, 2, 3\}$, $T = \{2, 3, 1\}$, $U = \{3, 2\}$

LHS \ RHS	S	T	U
S	$\subseteq$ T $\subset$ F	$\subseteq$ T $\subset$ F	$\subseteq$ F $\subset$ F
T	$\subseteq$ T $\subset$ F	$\subseteq$ T $\subset$ F	$\subseteq$ F $\subset$ F
U	$\subseteq$ T $\subset$ T	$\subseteq$ T $\subset$ T	$\subseteq$ T $\subset$ F

(exercise!).

Is set difference (\) commutative?

No.

$S \setminus U$ $\quad$ $U \setminus S$

In general $S \setminus T \neq T \setminus S$

$\{1\}$ $\{$ $S \setminus U$

$U \setminus S$

ϕ

Bidirectional Subset Relations: Programming

```
/* Return the set of positive elements from input. */
HashSet<Integer> allPositive(HashSet<Integer> input)
```

Formulate the `allPositive` method using a **set comprehension**.

input = {2, 3, -1, 4, -2}

allPositive(input) = {2, 3, 4}

{1, 2, 4} ✗

formulate

$$\text{allPositive}(\text{input}) = \{ x \mid \underbrace{x > 0}_{\text{not complete}} \land x \in \text{input} \}$$

Bidirectional Subset Relations: Programming

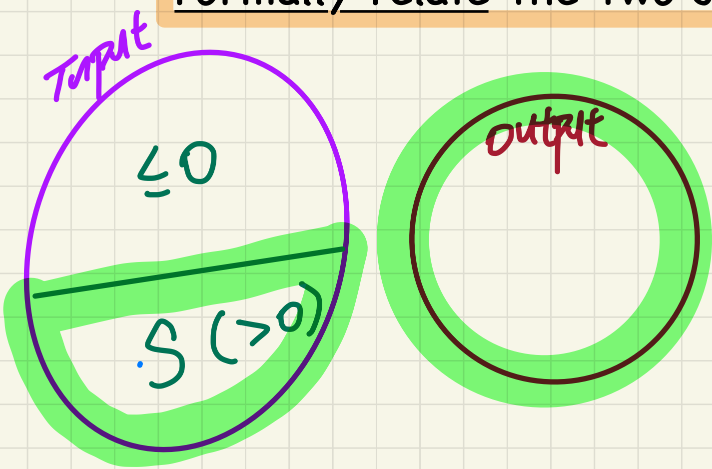
Post-Condition

```java
/* Return the set of positive elements from input. */
HashSet<Integer> allPositive(HashSet<Integer> input)
```

Say:

- S denotes the subset all positive elements from `input`.
- Set `output` denotes the return value from `allPositive`.

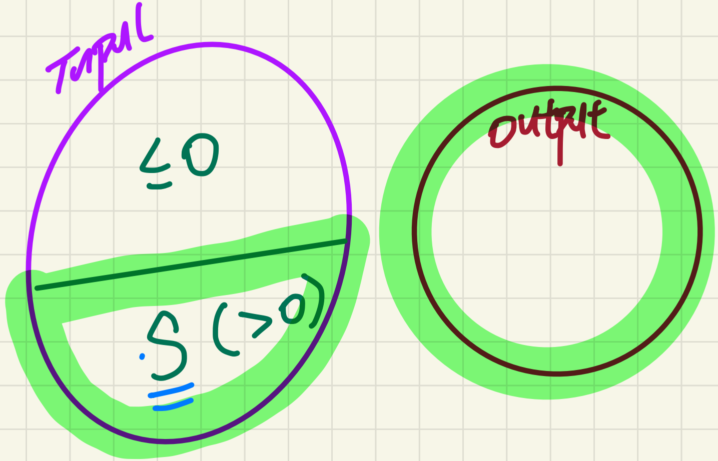
Formally relate the two sets S and **output**.



(N1)

$$\left.\begin{array}{l} (p1)\ output \subseteq S \\ (p2)\ S \subseteq output \end{array}\right\}\ S = output$$

- What if only p1 is required? e.g. $\emptyset$ can satisfy p1
- What if only p2 is required? e.g. output contains elements $\notin S$



$$\forall x \cdot x \in S \Leftrightarrow x \in \text{output}$$

valid on paper, but inconvenient to put into Rodin.

$x > 0$

(P1) $\forall x \mid x \in \text{output} \Rightarrow \boxed{x \in S}$

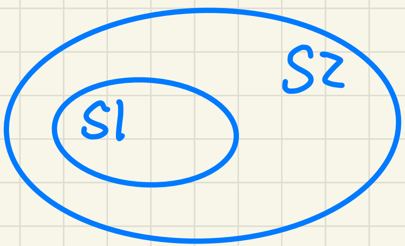
(P2) $\forall x \mid \underbrace{x \in \text{input} \land x > 0}_{x \in S} \Rightarrow x \in \text{output}$

$\emptyset \subset S$

True if $|S| > 0$

False if $|S| = 0$

$\because \emptyset \subset \emptyset$ F.



$S1 \subset S2$ T but $\not\Rightarrow$ $S1 = S2$

$S1 \subseteq S2$ T

Cardinality of Power Set: Interpreting Formula

flexible: e.g. how many subsets of card between l and u

- Calculate by considering subsets of various cardinalities.
- Calculate by considering whether a member should be included.

$$|\mathbb{P}(S)| = \underbrace{\binom{|S|}{0}}_{\text{\# of subsets of card 0}} + \underbrace{\binom{|S|}{1}}_{\text{\# subsets of card 1}} + \cdots + \underbrace{\binom{|S|}{|S|}}_{\text{\# maximum subset of card } |S|}$$

$\binom{|S|}{0} = 1$, $\binom{|S|}{1} = |S|$, $\binom{|S|}{|S|} = 1$

$\{a, b, c\}$

$\{a\}$, $\{b\}$, $\{c\}$

$$2^{|S|}$$

$S = \{a, b, c\}$

a	b	c	subset
0	0	0	$\emptyset$
0	0	1	$\{c\}$
⋮			
1	1	1	$\{a, b, c\}$

Lecture 1b

Review on Math: Relations

Set of Tuples

$|S_1 \times S_2 \times \cdots \times S_n| = |S_1| * |S_2| * \cdots * |S_n|$

cross product

multiplication

Given n sets $S_1, S_2, \ldots, S_n$, a ***cross/Cartesian product*** of theses sets is a set of n-tuples.

Each ***n-tuple*** $(e_1, e_2, \ldots, e_n)$ contains n elements, each of which a member of the corresponding set.

$\in S_1$ $\in S_2$

structure of each member in the resulting set

$$S_1 \times S_2 \times \cdots \times S_n = \{\boxed{(e_1, e_2, \ldots, e_n)} \mid e_i \in S_i \wedge 1 \leq i \leq n\}$$

Example: Calculate {a, b} ✗ {2, 4} ✗ {$, &}

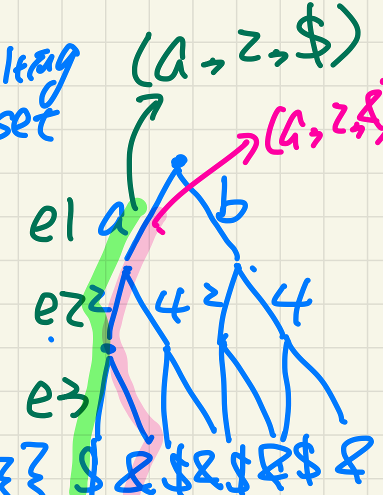
$\underbrace{\{a, b\}}_{S_1} \times \underbrace{\{2, 4\}}_{S_2} \times \underbrace{\{\$, \&\}}_{S_3}$

$= \{ \boxed{(e1, e2, e3)} \mid e1 \in \{a, b\} \wedge e2 \in \{2, 4\} \wedge e3 \in \{\$, \&\} \}$

$= \{ \quad \}$

8 members.

exercise!



Relation : set of pairs

e.g. Relation on $\{1, 2, 3\}$ and $\{a, b\}$

S_1 $\qquad$ S_2

- Is $(1, a)$ a relation on S_1 and S_2? No!

  not even a set!

- Is $\{(b, 2)\}$ a relation on S_1 and S_2?

  ↓ S_1 elements should come first in the ordered pair

- YES ~~Is~~ $\{(1, a), (3, b)\}$ a relation on S_1 and S_2?

- Minimum relation on S_1 and S_2? $\emptyset$ ⇝ empty relation!

- ✓ Maximum relation on S_1 and S_2? $S_1 \times S_2$

empty relation

$\{\{\}\}$

$\{\}$

① ×

② $\emptyset$

or

$\{\emptyset\}$

Given two sets $\underline{S}$ and $\underline{T}$:

- min relation : $\emptyset$
- max relation : $S \times T$
- set of All possible relations on S and T:

$$\left\{ \begin{array}{l} \underline{\emptyset} \text{ relation of card } 0 \\ \boxed{\begin{array}{l} \vdots \quad 1 \\ \vdots \quad 2 \\ \vdots \quad \vdots \\ \quad\quad |S| * |T| - 1 \end{array}} \\ \underline{S \times T} \text{ relation of card of card } |S| * |T| \end{array} \right\}$$

each member is a relation.